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Probability Densities and the Random Variable Transformation Theorem

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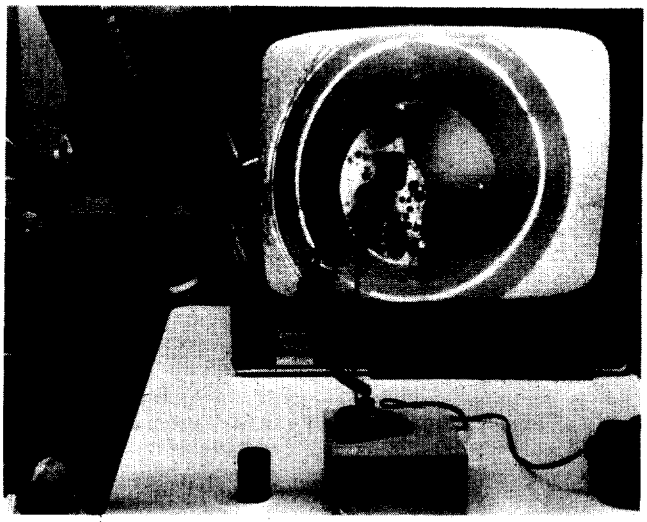

Fig. 1. Setup for the demonstration of black film formation in a lecture hall setting is illustrated. A cylindrical film holder, described in the text, appears in the lower foreground. Additional components are an illuminator, television camera, and a monitor. A partially black film is visible on the monitor screen.

convenient working distance. A typical image of a partially blackened film appears on the monitor screen. Films formed from this material are quite fluid. Convective movement of black areas is readily observed. Films can be supported in a vertical plane if desired, and films of larger area can readily be formed, though they tend to be less stable.

A convenient and portable hand-held demonstration appropriate for small groups can also be prepared using a glass or plastic vial fitted with a cap. The vial is partially filled with shampoo, then capped and inverted. Upon righting, then carefully removing the cap, a film will generally be formed across the mouth of the vial. The vial can then be dropped into a blind hole in a Delrin holder for viewing against a dark background under ambient light. Films formed in this way will not be flat because of a slight overpressure of air entrapped in the vial. The bottle in which the shampoo is supplied by the manufacturer may also be used as described. Our initial observation of black film formation by this material was made in this way.

ACKNOWLEDGMENT

We wish to thank R. L. Morandi for his capable technical assistance in the development of this demonstration.

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Probability densities and the random variable transformation theorem

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D. T. Gillespie[1,2] has recently derived and discussed a random variable transformation (RVT) theorem relating the joint probability densities of functionally dependent sets of random variables. This theorem may be compactly written in vector notation as

$$P_Y(\mathbf{y}) = \int d\mathbf{x}\, P_X(\mathbf{x})\delta(\mathbf{y} - \mathbf{f}(\mathbf{x})), \tag{1}$$

where $P_X(\mathbf{x})$ is the joint probability density for a set of random variables $\mathbf{X} = (X_1, X_2, ..., X_n)$, $P_Y(\mathbf{y})$ is the joint probability density for a second set of random variables $\mathbf{Y} = (Y_1, Y_2, ..., Y_m)$ which is functionally related to the first set by $\mathbf{Y} = \mathbf{f}(\mathbf{X})$, the delta function of a vector is defined in the usual way as the product of delta functions of its components, and $d\mathbf{x} = dx_1 dx_2 \cdots dx_n$. The probability densities are defined so that $P_V(\mathbf{v})\, d\mathbf{v}$ is the joint probability that the random variables $\mathbf{V}$ lie in the intervals $v_j < V_j < v_j + dv_j$.

Gillespie illustrates the utility of the RVT theorem by means of several simple applications. These applications show clearly that the theorem has considerable pedagogical value in providing a unified approach to a variety of problems in physics and statistics. It therefore seems worthwhile to observe that the RVT theorem is an immediate corollary of the simpler and more fundamental relation

$$P_Q(\mathbf{q}) = \langle \delta(\mathbf{q} - \mathbf{Q}) \rangle, \tag{2}$$

where $\mathbf{Q} = (Q_1, Q_2, ..., Q_k)$ is *any* set of random variables of interest and $\langle \cdots \rangle$ denotes an appropriately weighted average over all possible realizations of the underlying random system.

This definition of the average $\langle \cdots \rangle$ may seem rather vague and imprecise, but it is all that is needed for many purposes. Indeed, the vagueness is actually an advantage, for it lends a great deal of generality to Eq. (2). Many formal derivations can thereby be performed without ever specifying (or even identifying) a full set of basic or fundamental underlying random variables or parameters, upon which the variables Q_i of interest depend and from which the Q_i derive their randomness and statistical properties. It is, of course, implicit that the average $\langle \cdots \rangle$ is a linear operation (or more precisely a linear functional on functions of random variables).

Equation (2) is easily obtained from the basic relation

$$\langle F(\mathbf{Q}) \rangle = \int d\mathbf{q}\, P_Q(\mathbf{q}) F(\mathbf{q}), \tag{3}$$

where F is an arbitrary function. This relation, which is just Eq. (18) of Ref. 1 in vector notation, can in fact be regarded as the definition of the probability density $P_Q(\mathbf{q})$. Introducing the identity $\int d\mathbf{q}\, \delta(\mathbf{q} - \mathbf{Q}) = 1$ into the left member of Eq. (3), we obtain

$$\int d\mathbf{q} \langle \delta(\mathbf{q} - \mathbf{Q}) \rangle F(\mathbf{q}) = \int d\mathbf{q}\, P_Q(\mathbf{q}) F(\mathbf{q}), \tag{4}$$

and since this must hold for arbitrary $F(\mathbf{q})$ we may infer Eq. (2). Alternatively, setting $F(\mathbf{Q}) = \delta(\mathbf{q}' - \mathbf{Q})$ in Eq. (3) leads at once to Eq. (2) with the dummy variable $\mathbf{q}$ replaced by $\mathbf{q}'$.

It is a simple matter to deduce the RVT theorem from Eq. (2). We simply write

$$P_Y(\mathbf{y}) = \langle \delta(\mathbf{y} - \mathbf{Y}) \rangle = \langle \delta(\mathbf{y} - \mathbf{f}(\mathbf{X})) \rangle. \tag{5}$$

The latter average can be expressed in terms of $P_X(\mathbf{x})$ using Eq. (3), and this immediately yields the RVT theorem, Eq. (1).

Equation (2), in which $P_Q(\mathbf{q})$ is represented as a delta function averaged over an unspecified distribution of unspecified "internal" random variables, is well known to statistical physicists, but for some reason it rarely finds its way into the textbooks. This is unfortunate, for it is of considerable pedagogical and practical utility and should be in the repertoire of every practicing physicist. It is hoped that the present discussion will help to disseminate this basic relation among a wider circle of nonspecialists.

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Invariant form for the relativistic acoustic Doppler effect

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In a recent article Frankl gives an expression for the relativistic acoustic Doppler effect in three dimensions in terms of ordinary velocities relative to the rest frame of the acoustic medium.[1] He shows explicitly for the one-dimensional case that motion relative to the medium becomes meaningless when the signal travels at the speed of light. The present note extends this theorem to three dimensions by rewriting Frankl's expression in a relativistically invariant form.

In Frankl's notation the ratio of the period T'' of the signal relative to the receiver to the period T' of the signal relative to the source is

$$\frac{T''}{T'} = \frac{\gamma_v [1 - (v/s)\cos \alpha_S]}{\gamma_u [1 - (u/s)\cos \alpha_R]}, \tag{1}$$

where s, v, and u are the speeds of the signal, the source, and the receiver, respectively, relative to the medium; α_S and α_R are the angles made by v and u, respectively, with the line from the source to the receiver as measured in the rest frame of the medium; $\gamma_v \equiv (1 - v^2/c^2)^{-1/2}$, etc.; and c is the speed of light.

In order to put Eq. (1) into invariant form, introduce the proper velocities $V^\mu \equiv \gamma_v (c;v)$ of the source and $U^\mu \equiv \gamma_u (c;u)$ of the receiver. Let

$$P^\mu \equiv (E/c;P) = (c;c^2P/E)E/c^2 = (c;s)E/c^2 \tag{2}$$

be the energy-momentum four-vector of the signal, and let W^μ be the proper velocity of the medium. In the rest frame of the medium one has

$$W^\mu = (c;0), \tag{3}$$

and it is easily checked that Eq. (1) is equivalent to

$$T''/T' = K{\cdot}V/K{\cdot}U, \tag{4}$$

where

$$K^\mu \equiv P^\mu - (P{\cdot}P/P{\cdot}W)W^\mu \tag{5}$$

and, for example, $K{\cdot}V \equiv K^\mu V_\mu$ is the four-vector scalar product. Since the right-hand member of Eq. (4) contains invariants only, it can be evaluated in any reference frame.

If the signal consists of particles with rest mass m and (timelike) proper velocity $S^\mu \equiv \gamma_s (c;s)$ so that $E = m\gamma_s c^2$, then one has

$$P{\cdot}P = (s^2 - c^2)E^2/c^4 = -E^2/\gamma_s^2 c^2 = -m^2c^2. \tag{6}$$

If the signal travels with the speed of light, however, one has

$$P{\cdot}P = (s^2 - c^2)E^2/c^4 = 0 \tag{7}$$

and Eq. (4) reduces to

$$T''/T' = P{\cdot}V/P{\cdot}U. \tag{8}$$

This shows explicitly that for this case motion relative to